# Tutorial

Introduction to Micro- and Macroeconomics - ENS/EM Lyon

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**I. Consumers-workers.** Let us first consider a representative consumer-worker. She works $h$ hours out of $M$ (the maximum she can work) and gets an hourly wage $w > 0$. She consumes a quantity $x$ of the unique good of the economy, which has a unit price equal to $p > 0$.

The consumer-worker chooses her quantity of work $h \in [0, M]$ and her consumption $x$ so as to maximize her utility $u(x, h) = x^\gamma (M - h)^\delta$, with $(\gamma, \delta) \in ]0, 1[^2$.

1. (a) Explain why the worker's decision can be interpreted as a consumer problem with two goods, consumption $x$, whose price is $p$, and leisure $\lambda = M - h$, whose price is the opportunity cost $w$.

   (b) In this model, it is assumed that the consumer-worker has individually no influence on $p$ and $w$. Explain this assumption.

2. Compute the gradient of $u$. Give the economic meaning of its elements, and the economic interpretation of the assumption $\gamma, \delta > 0$.

3. Why can we infer that the consumer-worker is going to consume all her income? Give the corresponding budget constraint.

4. Compute the Hessian of $u$. Give the economic meaning of its elements, and the economic interpretation of the assumption $\gamma, \delta < 1$.

5. Using the budget constraint from question 3., express $u$ as a function of $h$ only. Check that previous assumptions on $\gamma$ and $\delta$ ensure the concavity of $u(h)$. Why does this matter?
   *Reminder: $f$ concave $\Leftrightarrow f'' \leq 0$.*

6. Write the first order condition of the maximization program of the consumer-worker, and express her optimal working time $h^*$ as a function of parameters. Deduce her optimal leisure $\lambda^*$.

7. Using her budget constraint, deduce her optimal consumption $x^*$ as a function of $p$ and $w$. Check that your results are consistent by looking at the values of $x^*$ and $h^*$ when $\gamma \to 0$ or $\delta \to 0$.

8. Using the insight of question 1.a and previous results, let us study income and substitution effects in the worker's consumption-leisure trade-off.
   (a) Write the consumer's 'expenditure' in consumption $x$ and leisure $\lambda$, and deduce that her 'income' in this problem is $wM$. Interpret this result.
   (b) Write the Slutsky equation for leisure $\lambda$, price $p$ and income $wM$. Compute the income effect and the sum of income and substitution effects. Which effect dominates? Is leisure a normal good or an inferior good?
   (c) Write the Slutsky equation for consumption $x$, wage $w$ and income $wM$. Compute the income effect and the sum of income and substitution effects. Which effect dominates? Is consumption a normal good or an inferior good?
   (d) Compute the elasticity of substitution between consumption and leisure $e = \frac{d\ln(x/\lambda)}{d\ln(w/p)}$. Are consumption and leisure complements or substitutes?
9. There are $N$ consumer-workers in the economy. Deduce the aggregate demand for consumption $D_C$ and the aggregate labor supply $H$.
10. Check that the previous result implies $pD_C = wH$. Explain why this relationship is expected.

**II. Firms.** The consumer-worker works in a representative firm, which uses capital and labor as production factors. Production $y$ is a function of the quantity of capital $k$ and the quantity of labor $l$, such that $y(k,l) = Ak^\alpha l^\beta$, with $A > 0$ and $(\alpha, \beta) \in ]0,1]^2$.

The firm chooses $k$ and $l$ so as to maximise its profit $\pi(k,l) = py(k,l) - rk - wl$, where $r > 0$ is the rental rate of capital.

1. How can you interpret the parameter $A$?
2. In this model, it is assumed that the firm has individually no influence on $p$, $w$ and $r$. Explain this assumption.
3. Compute the gradient of $y$. Give the economic meaning of its elements, and the economic interpretation of the assumption $\alpha, \beta > 0$.
4. Compute the Hessian of $y$. Give the economic meaning of its elements, and the economic interpretation of the assumption $\alpha, \beta < 1$.
5. Check that $y$ is homogeneous of degree $\alpha + \beta$. From now on, it is assumed that production has nonincreasing return to scales. Give the corresponding condition on $\alpha$ and $\beta$.
6. Why does it matter that $\pi$ be concave? Find a sufficient and necessary condition on $\alpha$ and $\beta$ for the concavity of $\pi$. How does this relate to previous assumptions?
   *Reminder: a function is concave if and only if its Hessian is negative-semidefinite. For a matrix of dimension 2, being negative-semidefinite means that both diagonal elements are nonpositive while the determinant is nonnegative (Sylvester's criterion).*

7. Write the first order conditions of the maximization program of the firm. Check that they imply that the marginal productivity of each factor is equal to its price (expressed in terms of the unique good), and explain this result. Then compute the average productivity of each factor.

8. Express the optimal capital ratio of the firm $\frac{k^*}{l^*}$ (where $^*$ denotes the situation at optimum) as a function of the relative price of factors $\frac{r}{w}$. Explain the behavior of the firm in case of factor price change.

9. Given any production $y$, the output elasticity to input $z$ writes $\frac{dy}{dz}\frac{z}{y}$. What does it describe? Compute the output elasticity to each factor, at optimum.

10. Infer from the first order conditions the factor shares $\frac{rk^*}{py^*}$ and $\frac{wl^*}{py^*}$, and then the profit share $\frac{\pi^*}{py^*}$.

    (a) Relate your result with that of preceding question. What happens in case of constant returns to scale?

    (b) How are factor share and average factor productivity related (question 7.)? Does this contradict the fact that factors are paid at their marginal productivity?

11. Using questions 7. and 8., compute the elasticity of substitution between capital and labor. Explain what this notion describes and relate your result with that of preceding question.

12. Let us assume that the firm produces an arbitrary quantity $q$ while keeping production factors at their optimal ratio given in question 8. Using this ratio, solve $q = y(k, l)$ to show that $k(q)$, $l(q)$, and finally the cost function $c(q) = rk(q) + wl(q)$ are linear in $q^{\frac{1}{\alpha+\beta}}$. What is the slope of the marginal cost $c'(q)$? What happens in case of constant returns to scale? Relate this result with that of question 10a.

From now on, we assume strictly decreasing returns to scale.

13. Using the optimal capital ratio of question 8., solve the system of question 7. to compute the firm demands for capital and labor $k^*$ and $l^*$, and finally production $y^*$, as functions of prices.

14. There are $J$ firms in the economy.

    (a) Express the aggregate supply of goods $Y$, the aggregate demand for capital $K$ and the aggregate demand for labor $L$ as functions of $y^*$ (use factor shares).

    (b) Using the results from question 13., express $Y$ and $L$ as a function of prices.

**III. Capitalists.** Each firm is owned by a capitalist, whose objective is to maximize her wealth. The capitalist gets the profit of the firm, $\pi^*$. Let us note that the capital of her firm is not necessarily hers, since it is leased by the firm on the capital market. However, each capitalist invests all her revenues in the capital market.

1. Investing in the capital market is a two-step process. In the first one, the capitalist uses the profit of the firm to buy an amount $c_0$ of investment goods on the goods market. Express $c_0$ as a function of $\pi^*$.
2. In a second step, she leases these investment goods to firms at the rental price $r$, and gets from this operation a nominal dividend $d_1$. Express $d_1$ as a function of $c_0$.
3. Explain why $c_0$ can be interpreted as the initial real wealth of the capitalist. Express her real wealth $c_1$, after getting $d_1$, as a function of $c_0$.
4. Now, the capitalist entirely reinvests $d_1$ on the capital market. What is the nominal dividend $d_2$ that she gets from this operation? What is her real wealth $c_2$ now? Express both as functions of $c_0$.
5. This operation is repeated infinitely. Explain why this is consistent with the capitalist' objective.
6. For any $n \in \mathbb{N}$, express $d_{n+1}$ as a function of $d_n$, then $d_n$ as a function of $c_0$, and finally $d_n$ as a function of $\pi^*$.
7. For any $n \in \mathbb{N}$, express $c_{n+1}$ as a function of $c_n$ and $d_{n+1}$, then $c_n$ as a function of $c_0$, and finally $c_n$ as a function of $\pi^*$.
8. Assuming $r < p$, compute the capitalist' final real wealth $c = \lim c_n$ as a function of $\pi^*$.
9. Using the result from question II.10, write $c$ as a function of $y^*$.
10. The aggregated supply of capital $C$ is equal to the sum of all capitalists' investments (and thus of all capitalist's wealth, since all their wealth is invested). It is also equal to the aggregated demand for investment goods $D_I$. Give $C$ and $D_I$ as functions of $y^*$.

**IV. General equilibrium.** The economy is made of three markets : the capital market, the labor market and the goods market. Prices adjust such that the aggregate demand is equal to the aggregate supply on each market at equilibrium.

1. (a) Make a table with two rows (supply and demand) and three columns (one for each market). Indicate in each cell who are the suppliers or demanders (workers, firms, capitalists).

   (b) There are nine flows in the economy: consumption goods, investment goods, consumption expenditures, investment expenditures, labor, capital, wages, dividends and profits. Draw two graphs where these flows are represented as arrows: one graph for real flows and one graph for nominal flows, each of them with three nodes (workers, firms and capitalists).

(c) Which flow does not correspond to any market?

2. Write the equilibrium condition on the capital market.
   (a) Using questions II.14 and and III.10, infer the equilibrium real rental rate of capital $\rho^* = r^*/p$.
   (b) Check that the assumption of question III.8, $r < p$, is verified.
   (c) Show that the market equilibrium condition condition can be written $r(1-\beta)Y = p\alpha Y$.
   (d) Comment the effect of $\alpha$ and $\beta$ on $\rho^*$.

3. Write the equilibrium condition on the labor market.
   (a) Using the results from previous question, question I.9 and II.14b, deduce the equilibrium real wage $\omega^* = w^*/p$.
   (b) Comment the effect of the ratio of workers to firms $N/J$ on the real wage.

4. The aggregated demand on the goods market $D$ is the sum of the aggregated demands for consumption goods and investment goods: $D = D_C + D_I$. Write the equilibrium condition on the goods market.
   (a) Show that regardless of this equilibrium, $D_C = \beta Y$ and $D_I = (1-\beta)Y$. Is $p$ definite in this model?
   (b) Why is this expected?

5. Using results from questions II.14b, IV.2 and IV.3, compute the production level at equilibrium.
   (a) Compute $\frac{\partial Y}{\partial p}$, and explain your result.
   (b) Explain why $Y$ is proportional to $A^{\frac{1}{1-\alpha}}$ while $y(k,l)$ is proportional to $A$.
   (c) Deduce the wealth of each capitalist $c^*$. Comment the effect of the ratio of workers to firms $N/J$ on the capitalist' wealth.

6. Suppose that the government finds the equilibrium wage too low, and thus introduces a minimum wage above the equilibrium wage.
   (a) Suppose first that the government introduces a nominal minimum wage $\underline{w}$, such that $w \geq \underline{w} > w^*$. What happens then in our model?
   (b) Suppose then that the government indexes $\underline{w}$ on the price of goods, which amounts to introducing a real minimum wage $\underline{\omega}$ such that $\omega \geq \underline{\omega} > \omega^*$. What happens then in our model?

7. Suppose that the government decides to redistribute part of the wealth of capitalists to workers. To do this, the government taxes the wealth of each capitalist at rate $t \in [0, 1[$ to fund a $s\%$ increase in wages, so that workers get $(1+s)w$ while firms still pay them $w$.

(a) What is the effect of this policy on labor supply? Explain why the government prefers wage subsidies to an unconditional wealth transfer to workers.

(b) What is the effect of this policy on labor demand, capital demand, and finally supply of goods, as functions of prices? Explain why the government prefers to tax wealth rather than transactions on the capital market.

(c) What is the effect of this policy on capital supply? Write the new equilibrium on the capital market. Using question IV.2, infer the new equilibrium rental rate of capital $\hat{r}$ as a function of $r^*$ and $t$. Explain the sign of $\hat{r} - r^*$.

(d) Using question IV.3, infer from the equilibrium on the labor market the new (pre-subsidy) equilibrium wage $\hat{w}$ as a function of $w^*$ and $t$.

(e) Using question IV.5, deduce the new production at equilibrium $\hat{y}$ as a function of $y^*$ and $t$. Explain the sign of $\hat{y} - y^*$ and the role of $\alpha$.

(f) Infer the tax rate $t = \frac{s\hat{w}\hat{H}}{p\Phi}$ from the budget constraint of the government, where $\Phi$ is the pre-tax wealth. Explain why the government may prefer to tax wealth rather than profits. Would taxing profits change the relationship between $\hat{y}$ and $y^*$?

(g) Using the pre-tax labor share and results from questions IV.4a and IV.7c, show that the government's budget constraint implies $t = \frac{s\beta}{1-\beta}$. Explain why this relationship is consistent.

(h) Show that the redistribution scheme benefits workers if and only if $t(1-t)^\alpha > \beta$. Study the fonction $f(t) = \ln t + \alpha \ln(1-t) - \ln \beta$. Show that workers get the maximum benefit for $t = \frac{1}{1+\alpha}$ if $f(\frac{1}{1+\alpha}) > 0$, and $t = 0$ otherwise.

(i) Show that this condition is more likely to be met if $\alpha$ and $\beta$ are low. Explain this result.

(j) Compute $f$ for $\alpha = 0.4$ and $\beta = 0.5$, and then for $\beta = 0.4$. Compute the corresponding optimal tax and subsidy rates. Comment your results.

8. What are the limitations of this model? What assumptions would you change to improve it?