Where does the Cobb-Douglas function come from?

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Abstract

The choice of the Cobb-Douglas function (product of power functions) may seem, at first glance, somewhat arbitrary. So why is it so widely used in economics? Simply because it is the *only* function with constant elasticity of output with respect to any input, a stronger property than CES.

Consider an objective function $F(X)$, $X = (x_1, ..., x_n) \in \mathbb{R}^n$. Let us define the function ϕ_i as the multiplicative change in F resulting from a multiplicative change $\lambda > 0$ in x_i:

$$\phi_i(\lambda, X) = \frac{F(x_1, ..., \lambda x_i, ...x_n)}{F(X)}.$$

We consider a continuously differentiable F, and thus ϕ_i as well for nonzero values of F.

The property we are interested in is the restriction $(\forall i)$

$$\phi_i(\lambda, X) = \phi_i(\lambda). \tag{1}$$

It means that a multiplicative change in x_i has a constant effect on F, whatever the values of its arguments (including x_i). The economic translation of that property is *constant elasticity* of output with respect to any input[1].

This property is useful for two reasons: it removes scale effects at the input level[2], and it makes it possible to analyze the effect of each input independently of each other:

$$\frac{\partial F}{\partial x_i}(X) = \lim_{\lambda \to 1} \frac{(\phi_i(\lambda) - 1)F(x)}{(\lambda - 1)x_i}$$

so

$$\frac{d \ln F(X)}{d \ln x_i} = \lim_{\lambda \to 1} \frac{(\phi_i(\lambda) - 1)}{(\lambda - 1)}$$

[1]Not to be confused with the weaker property of *constant elasticity of substitution* (CES). Constant output elasticity implies CES = 1, as shown below.

[2]Scale effects may still appear when λ is applied to all inputs (returns to scale).

(which is the constant elasticity).

Now, if (1) is true, then we can rewrite

$$F(x) = A\prod_i \phi_i(x_i), \tag{2}$$

with $A = F(1, ..., 1)$.

Then, what about theses functions ϕ_i?

Well, according to equation (2),

$$F(x_1, ..., \lambda x_i, ...x_n) = \phi_1(x_1)...\phi(\lambda x_i)...\phi_n(x_n)$$

which, combined with equation (1) for X such that $F(X) > 0$, imply that $\forall \lambda, x_i > 0$,

$$\phi(\lambda x_i) = \phi(\lambda)\phi(x_i). \tag{3}$$

Now, let us note $a = \ln \lambda$, $b = \ln x_i$, and $f_i(x) = \ln \phi_i(e^x)$. Then (3) rewrites

$$f_i(a + b) = f_i(a) + f_i(b). \tag{4}$$

In particular, $f_i(a) = f_i(a) + f_i(0)$, so $f_i(0) = 0$.

Equation (4) also imply that $\forall n \in \mathbb{N}$,

$$f_i(na) = nf_i(a). \tag{5}$$

Equation (5) applies to $a = \frac{y}{n}$, for whatever $y > 0$, which gives

$$f_i(y) = nf_i\left(\frac{y}{n}\right). \tag{6}$$

We assumed ϕ_i to be continuously differentiable, so f_i is so as well, and equation (6) implies

$$f_i'(y) = f_i'\left(\frac{y}{n}\right) \underset{n\to\infty}{\longrightarrow} f_i'(0) = \alpha_i \in \mathbb{R}. \tag{7}$$

Then we can integrate equation (7):

$$\alpha_i x = \int_0^x f_i'(y)dy = f_i(x) - f_i(0) = f_i(x),$$

which is equivalent to

$$\phi_i(z) = z^{\alpha_i}.$$

Finally, we find that

$$F(X) = A\prod_i x_i^{\alpha_i},$$

that is the Cobb-Douglas function.

Note 1: Constant elasticity of output corresponds to fixed factor shares.

The producer program implies that each input is paid at its marginal productivity: $\frac{\partial F}{\partial x_i} = p_i$. The factor share thus writes

$$\frac{p_i x_i}{F(X)} = \frac{\partial F}{\partial x_i} \frac{x_i}{F(X)} = \alpha_i \in \mathbb{R}_+^*$$

the constant output elasticity to input i.

Note 2: Constant elasticity of output implies elasticity of substitution = 1.

Constant output elasticity to any input implies

$$\frac{F_{x_j}}{F_{x_i}} = \frac{x_j F_{x_j}}{F(X)} \frac{F(X)}{x_i F_{x_i}} \frac{x_i}{x_j} = \frac{\alpha_j}{\alpha_i} \frac{x_i}{x_j}. \tag{8}$$

Thus

$$\frac{d \ln \frac{F_{x_j}}{F_{x_i}}}{d \ln \frac{x_i}{x_j}} = 1.$$

Exercise left to the reader: does $CES = 1$ imply constant elasticity of output?