Tutorial II - Corrections

Mathematics for economists - ENSL Premaster

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Exercise 3.

1. (a) Show that a point x is in the closure of S if and only if every neighbourhood of x contains at least one point of S.

Correction.

Lemma: ${}^C\bar{S}$ is the union of all open sets of empty intersection with S.

Let (A_i) be all closed sets such that $S \subset A_i$. By definition, $\bar{S} = \bigcap_i A_i$, thus by De Morgan's laws ${}^C\bar{S} = \bigcup_i ({}^C A_i)$. As $(\forall i)$ $S \subset A_i$, $({}^C A_i) \cap S = \emptyset$. ${}^C\bar{S}$ is thus an (infinite) union of open sets each of empty intersection with S.

Let us now assume that there is an open set O such that $O \cap S = \emptyset$ and $O \not\subset ({}^C\bar{S})$. $O \cap S = \emptyset$ implies that $\forall x \in S$, $x \notin O$, ie. $x \in ({}^C O)$. Thus $S \subset ({}^C O)$. As ${}^C 0$ is a closed set (complementary of O) containing S, it is thus also containing the closure of S (intersection of all closed sets containing S): $\bar{S} \subset ({}^C O)$.
The assumption $O \not\subset ({}^C\bar{S})$ implies that $\exists x \in O$, such that $x \notin ({}^C\bar{S})$, ie. $x \in \bar{S}$. According to the previous result, then $x \in ({}^C O)$, ie. $x \notin O$, which is absurd. Thus every open O such that $O \cap S = \emptyset$ is such that $O \subset ({}^C\bar{S})$.

($\Longrightarrow$) Let us consider $x \in \bar{S}$, and let us assume that there is a neighborhood V_0 of x such that $V_0 \cap S = \emptyset$. This implies that there is an open ball B_0 centered in x such that $B_0 \subset V_0$. B_0 is thus an open set of empty intersection with S, and thus included in ${}^C\bar{S}$, according to the lemma. Thus $x \notin \bar{S}$, which contradicts the hypothesis. Thus V_0 does not exist.

($\Longleftarrow$) Let us consider x such that every neighborhood V of x is such that $V \cap S \neq \emptyset$. Now let us assume that $x \notin \bar{S}$, ie. $x \in ({}^C\bar{S})$. According to the lemma, $({}^C\bar{S})$ is an (infinite) union of open sets each of empty intersection with S, and thus an open set of empty intersection with S. There is then a neighborhood V_0 of x included in ${}^C\bar{S}$ (as it is open), and thus such that $V_0 \cap S = \emptyset$. This contradicts the hypothesis, thus $x \in \bar{S}$.