TD 1

Mathématiques pour économistes (prémaster)

20 Septembre 2023

Exercise 1.

Obvious eigenvalues

Each of these matrices has an obvious eigenvalue : which one and why ?

$$A = \begin{pmatrix} 1 & 0 & 5 & -9 \\ 7 & 4 & 0 & 5 \\ 2 & 0 & -3 & -1 \\ -1 & 0 & 2 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 5 & -9 \\ 3 & 1 & 0 & 2 \\ 0 & 0 & 7 & 0 \\ -1 & 3 & 2 & 8 \end{pmatrix}$$

Exercise 2.

Matrix reduction

Find the eigenvalues and associated eigenvectors of these matrices. Are they diagonalizable ? If so, find a base of eigenvectors.
Remember that you are not limited to only one method.

$$A = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} -2 & 1 & -1 \\ -1 & 0 & 1 \\ -2 & 2 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & -1 & 0 \\ 4 & -4 & 0 \\ 2 & -1 & -2 \end{pmatrix} \qquad E = \begin{pmatrix} -3 & -2 & -4 \\ 1 & -3 & 1 \\ 2 & 1 & 3 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Exercise 3.

Triangular matrix

An upper (lower) triangular matrix has only zeros under (above) the diagonal, eg.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_{nn} \end{pmatrix}$$

1. What are the eigenvalues of a triangular matrix ? Prove it without using the determinant.

2. Compute $det(\lambda I_n - A)$, with $\lambda \in \mathbb{R}$, and show again the previous result.

3. Derive a sufficient condition for diagonalizability of triangular matrices.

4. Assume that a matrix (of any kind) has only one eigenvalue (of any multiplicity). In which case is it diagonalizable ? Conclude for triangular matrices.

Exercise 4.

Block diagonal matrix

Let $(M_i) \in \prod_i \mathcal{M}_{p_i,q_i}$ be n matrices of any dimensions, where $i \in [\![1, n]\!]$, and

$$M = \begin{pmatrix} M_1 & 0 & ... & 0 \\ 0 & M_2 & ... & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & ... & M_n \end{pmatrix}$$

the resulting block diagonal matrix. Which rank is M ? Prove it. *Hint : begin with n=2.*

Exercise 5.

Gaussian elimination

Let us consider the following square matrices, of dimension larger than 3.

$$A = \begin{pmatrix} 1 & 1 & ... & 1 \\ 0 & 1 & ... & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & ... & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & ... & 1 \\ \vdots & \ddots & \vdots \\ 1 & ... & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 & 0 & ... & 0 \\ 0 & 1 & 1 & ... & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & ... & 0 & 1 & 1 \\ 0 & ... & 0 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & ... & 0 & 1 \\ 0 & 1 & 0 & ... & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \vdots & \ddots & 1 & 0 \\ 1 & 0 & ... & 0 & 1 \end{pmatrix}$$

Find the eigenvalues and associated eigenvectors of these matrices. Are they diagonalizable ? Are they invertible ? If so, compute the inverse using Gaussian elimination.

Exercise 6.

Similarity

Consider two square matrices A and B.

1. Assume that A and B are similar. Do they necessarily have the same rank ? The same trace ? The same eigenvalues ? The same eigenvectors ? Explain.

2. Discuss the similarity of A and B in the following cases.
– A and B are diagonalizable and have the same eigenvalues (with same algebraic multiplicities).
– A and B are diagonalizable and do not have the same eigenvalues and same algebraic multiplicities.
– A is diagonalizable and B is not.
– A and B are not diagonalizable.

Exercise 7.

Cofactors and inverse matrix formula

Let $A = [a_{ij}]$ be a square matrix. The (i, j)-minor m_{ij} is the determinant of the submatrix of A resulting from the deletion of line i and column j.
The (i, j)-cofactor is then defined as $c_{ij} = (-1)^{i+j} m_{ij}$, and the cofactor matrix (also called comatrix) of A as the matrix of its cofactors : $\text{Com}A = [c_{ij}]$.
Let us recall that cofactors can be used to compute the determinant using the following formula : $\forall (i, j)\ \ \det A = \sum_k a_{ik} c_{ik} = \sum_k a_{kj} c_{kj}$.

1. Show that all diagonal elements of ${}^{\text{T}}\text{Com}A \times A$ are equal to $\det A$.

2. Let us define B_{ij} as the matrix obtained from A when column i is replaced by column j. What is the value of $\det B_{ij}$?

3. Show that for all (i, j), the nondiagonal element (i, j) of ${}^{\text{T}}\text{Com}A \times A$ is equal to the diagonal element i of ${}^{\text{T}}\text{Com}B_{ij} \times B_{ij}$.

4. Using the three preceding questions, conclude that nondiagonal elements of ${}^{\text{T}}\text{Com}A \times A$ are zeros, and thus that ${}^{\text{T}}\text{Com}A \times A = \det A \cdot I$.

5. Show that $A \times^{\text{T}} \text{Com}A = \det A \cdot I$. *Hint : it should not take more than one line in the case* $\det A \neq 0$.

6. Conclude that if $\det A \neq 0$, A is invertible and $A^{-1} = \frac{1}{\det A}{}^{\mathrm{T}}\mathrm{Com}A$.

7. Check that this result is consistent with the inverse matrix determinant formula : $\det A^{-1} = 1/\det A$.

8. Derive a formula to invert a square matrix of dimension 2. Check it.

Exercise 8.

Using the rank–nullity theorem

Let us recall for this exercise the Rank–nullity theorem : let E and F be vector spaces, with E finite-dimensional. $\forall f \in \mathcal{L}(E, F)$,

$$\dim(\mathrm{Ker} f) + \mathrm{rk} f = \dim E.$$

Now, let E be a finite-dimensional vector space, and let us consider endomorphisms $(f, g) \in \mathcal{L}(E)^2$.

1. Prove that $(\mathrm{rk} f = \mathrm{rk} f^2) \Longleftrightarrow (E = \mathrm{Ker} f \oplus \mathrm{Im} f)$.
Hint : prove first that $(\mathrm{rk} f = \mathrm{rk} f^2) \Longleftrightarrow (\mathrm{Ker} f = \mathrm{Ker} f^2)$.

2. Assume that $f \circ g \circ f = f$ and $g \circ f \circ g = g$. Show that $E = \mathrm{Ker} f \oplus \mathrm{Im} g = \mathrm{Ker} g \oplus \mathrm{Im} f$.
Hint : apply the preceding result to $g \circ f$.