Tutorial II

Mathematics for economists - ENSL Premaster

Exercise 1.

Discuss the openness, closeness and compactness of each of the following subsets in their respective topological space and give their interior and closure.

- $[0,1]$ in $\mathbb{R}$
- $]0,1[$ in $\mathbb{R}$
- $[0,1[$ in $\mathbb{R}$
- $\mathbb{R}$ in $\mathbb{R}$
- $\emptyset$ in $\mathbb{R}$
- $\emptyset$ in $\emptyset$
- $\{a\}$ in $\mathbb{R}$, $a \in \mathbb{R}$
- $\mathbb{R}^*$ in $\mathbb{R}$
- $\mathbb{R}_+$ in $\mathbb{R}$
- $\mathbb{R}_+^*$ in $\mathbb{R}$
- $\mathbb{N}$ in $\mathbb{R}$
- $\{1/n \mid n \in \mathbb{N}^*\}$ in $\mathbb{R}$
- $\mathbb{Q}$ in $\mathbb{R}$
- $\mathbb{R}\backslash\mathbb{Q}$ in $\mathbb{R}$
- $\mathbb{R} \times \{0\}$ in $\mathbb{R}^2$
- $\mathbb{R}^p \times \prod_{i=p+1}^{n}\{a_i\}$ in $\mathbb{R}^n$, $p < n$
- $]0,1] \times \{1\}$ in $\mathbb{R}^2$
- $\{x, y \mid x + y = 0\}$ in $\mathbb{R}^2$
- $\{x, y \mid x + y > 0\}$ in $\mathbb{R}^2$
- $\{x, y \mid x = 0 \vee y = 0\}$ in $\mathbb{R}^2$
- $\{x, y | x^2 + y^2 = 1\}$ in $\mathbb{R}^2$
- $\{x, y | x^2 + y^2 = 1\}$ in $\mathbb{R}^2$
- $\{x, y | x^2 y^2 > 1\}$ in $\mathbb{R}^2$
- $\{x, y, z | x^2 + y^2 + z^2 < 1\}$ in $\mathbb{R}^3$

Exercise 2.

Let E be a vector space equipped with a norm denoted by $||.||$, and let $(A_i)_{i\in\mathbb{N}}$ be a sequence of sets in E: $\forall i \in \mathbb{N}$, $A_i \subset E$.

1. (a) Show that if all A_i are open, their union $\bigcup_{i\in\mathbb{N}} A_i$ is also open.
   (b) Show that even if all A_i are open, their intersection $\bigcap_{i\in\mathbb{N}} A_i$ is not necessarily open. *Hint: find a counterexample in $E = \mathbb{R}$.*

(c) Show that if there is $I \subset \mathbb{N}$, with I finite, such that $\forall i \in I$, A_i is open, then $\bigcap_{i\in I} A_i$ is also open.

2. (a) Show that if all A_i are closed, their intersection $\bigcap_{i\in\mathbb{N}} A_i$ is also closed.
   (b) Show that even if all A_i are closed, their union $\bigcup_{i\in\mathbb{N}} A_i$ is not necessarily closed.
   (c) Show that if there is $I \subset \mathbb{N}$, with I finite, such that $\forall i \in I$, A_i is closed, then $\bigcup_{i\in I} A_i$ is also closed.

3. (a) Show that if all A_i are compact, their intersection $\bigcap_{i\in\mathbb{N}} A_i$ is also compact.
   (b) Show that even if A_i are compact, then the union $\bigcup_{i\in\mathbb{N}} A_i$ is not necessarily compact.
   (c) Show that if there is $I \subset \mathbb{N}$, with I finite, such that $\forall i \in I$, A_i is compact, then $\bigcup_{i\in I} A_i$ is also compact. *Hint: draw the problem in* $\mathbb{R}$ *or* $\mathbb{R}^2$.

4. (a) Show that the union of interiors is included in the interior of the union.
   (b) Show that this inclusion can be strict.

5. (a) Show that the interior of the intersection is included in the intersection of interiors.
   (b) Show that this inclusion is an equality in the finite case.
   (c) Find a counter-example in the infinite case.

6. (a) Assume that all A_i are non-empty, and such that (A_i) is non-increasing (i.e. $\forall i \in \mathbb{N}$, $A_{i+1} \subseteq A_i$)[1]. Show that their intersection $\bigcap_{i\in\mathbb{N}} A_i$ is not necessarily non-empty.
   (b) Show that if all A_i are compact, non-empty, and such that (A_i) is non-increasing, then their intersection $\bigcap_{i\in\mathbb{N}} A_i$ is non-empty.
   To this purpose, let us admit the following proposition: if $(S_i)_{i\in\mathbb{N}}$ *is a sequence of closed subsets of the compact set* S*, such that* $\bigcap_{i\in\mathbb{N}} S_n = \emptyset$*, then* $\exists I \subset \mathbb{N}$*, with* I *finite, such that* $\bigcap_{i\in I} S_i = \emptyset$ *(if the whole sequence has an empty intersection, then a finite family with empty intersection can be extracted from the sequence).*

Exercise 3.

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1. For $S \subset U$, let us define $A \equiv \{X \subset U | S \subset X\}$ and $B \equiv \{X \subset U | X \cap S = \emptyset\}$. Let us note $\alpha \equiv \bigcap_{X\in A} X$ and $\beta \equiv \bigcup_{X\in B} X$. Show that $\beta = U \backslash \alpha$.

2. Show that the complement of the closure of S is the union of all open sets that have no intersection with S.

[1] Beware: it is standard in English to use 'increasing' in the sense of 'strictly increasing' and 'non-decreasing' in the sense of '(non-strictly) increasing' (and 'decreasing'/'non-increasing' instead of 'strictly decreasing'/'decreasing'). Note also that while $\subset$ is often used to denote non-strict inclusion, it may also be restricted to strict inclusion, in which case non-strict inclusion is denoted by $\subseteq$.

Exercise 4.

Hint: draw an example in $\mathbb{R}$ or $\mathbb{R}^2$ for each question.

1. (a) Show that x is in the closure of S if and only if every neighbourhood of x contains at least one point of S. *The result from Exercise 3 may be useful.*

   (b) Prove that if x is in the closure of S if and only if x is the limit of a sequence taking its values in S.

2. (a) Show without using sequences that the closure of an open ball of $\mathbb{R}^n$ is the closed ball of same center and radius.

   (b) Construct the closure of an open ball of $\mathbb{R}^n$ with sequences.

3. A point x is said to be an *isolated point* of the set S when $x \in S$ and there exists a neighborhood of x in which x is the only point of S.

   Show that x is an isolated point of S if and only if the singleton $\{x\}$ is an open set in S.

4. A point x is said to be an *accumulation point* of the set S when every neighborhood of x contains at least one point of S which is not x.

   Show that the set of accumulations points of S is equal to the subset of non-isolated points of the closure of X.

Exercise 5.

A *Cauchy sequence* is a sequence whose elements are getting closer and closer to each other. Formally, in a vector space E equipped with a norm denoted by $||.||$, a Cauchy sequence (u_n) can be characterized as following:

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall n, p \geq N, \quad ||u_n - u_p|| < \varepsilon.$$

1. Show that every convergent sequence is a Cauchy sequence.

2. Show that every Cauchy sequence is bounded.

3. Show that every subsequence $(u_{n_k})_k$ of a Cauchy sequence (u_n) is also a Cauchy sequence.

4. Show that if a Cauchy sequence has a convergent subsequence, the whole sequence converges toward the subsequential limit, and that a Cauchy sequence has at most one subsequential limit.

5. A normed space in which every Cauchy sequence converges is said to be *complete*[2].

   (a) The Bolzano-Weierstrass theorem states that a space is compact if and only if every bounded sequence has at least one convergent subsequence. Is $\mathbb{R}$ complete? Is $\mathbb{R}^n$ complete?

   Let us remark that Cauchy convergence and its properties are not restricted to vector spaces[3]. In the next questions, we also consider subsets of normed vector spaces in which these notions are valid. We call them subspaces in the sense that they are equipped with the space norm, even though they are not necessarily proper vector subspaces[4]. This remark is for the sake of rigor and does not matter for the proofs.

   (b) Show that every closed subspace of a complete space is complete.

   (c) Show that every complete subspace is closed.

   (d) Show that $\mathbb{N}$ is complete and $\mathbb{Q}$ is not.

Exercise 6.

Consider a segment, of which one removes the middle third; and then, the middle thirds of both remaining segments; and then, the middle thirds of the four remaining segments; and so on. Let us study what happens if this process is repeated *endlessly*.

Formally, let us define the "middle third removing" operator Θ over the set of real intervals:

$$\Theta : [a, b] \longrightarrow \left[a, a + \frac{b-a}{3}\right] \cup \left[b - \frac{b-a}{3}, b\right],$$

and let it be applied to an union of intervals, such that $\Theta(\bigcup_i I_i) = \bigcup_i \Theta(I_i)$. Let us then consider the sequence of sets (A_n) defined by $A_0 = [0, 1]$ and $\forall n \in \mathbb{N}$, $A_{n+1} = \Theta(A_n)$.

1. Draw A_0, A_1, A_2 and A_3.

2. For a given $n \in \mathbb{N}$, discuss the openness, closeness and compactness of A_n in $\mathbb{R}$ and give its interior and closure (it is not necessary to finnd the exact formula for A_n).

3. Show that (A_n) is (strictly) decreasing, in the sense that $\forall n \in \mathbb{N}$, $A_{n+1} \subseteq A_n$ and $A_n \nsubseteq A_{n+1}$[5].

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[2] Let us recall that a sequence is said to be convergent if and only if it has a limit *in the space in which the sequence is defined*.

[3] They are actually defined in *metric spaces*. A metric space is merely a non-empty set equipped with a distance. In a normed vector space, the distance d is given by the norm: $d(x, y) = ||x - y||$.

[4] They are still metric (sub)spaces.

[5] Same remark as in footnote 1.

4. Show that $\forall (p,q) \in \mathbb{N}^2$, $p < q$, $A_q = \bigcap_{n \in [\![p,q]\!]} A_n$.

5. The Cantor set K can be defined as $K = \bigcap_{n \in \mathbb{N}} A_n$. Discuss its closeness and compactness in $\mathbb{R}$, whether it is empty or not, and give its closure. *Use the results from exercise 2.*

6. Show that $\forall n \in \mathbb{N}$, A_n is the union of 2^n intervals, each of length $(\frac{1}{3})^n$, and that $\int_{A_n} \mathrm{d}x = (\frac{2}{3})^n$. Then show that $\int_K \mathrm{d}x = 0$.

7. Show that the interior of K is empty. Deduce whether K is open in $\mathbb{R}$ or not.

8. Show that K have no isolated points and that all points of K are accumulation points of K. *See exercise 3, question 3 and 4.*

Exercise 7.

1. (a) Let A and B be two open sets of $\mathbb{R}$. Show that $A \times B$ is an open set of $\mathbb{R}^2$. *Hint: draw it.*

   (b) Let A and B be two sets of $\mathbb{R}$, such that $A \times B$ is an open set of $\mathbb{R}^2$. Show that A and B are open in $\mathbb{R}$. *Hint: draw it.*

2. (a) Let A be an open set of $\mathbb{R}^n$ and B an open set of $\mathbb{R}$. Show that $A \times B$ is an open set of $\mathbb{R}^{n+1}$.

   (b) Let A be a set of $\mathbb{R}^n$ and B a set of $\mathbb{R}$, such that $A \times B$ is an open set of $\mathbb{R}^{n+1}$. Show that A and B are respectively open in $\mathbb{R}^n$ and $\mathbb{R}$.

3. (a) Let $(A_i)_{i \in [\![1,n]\!]}$ be n open sets of $\mathbb{R}$, with $n \in \mathbb{N}^*$. Show by induction that $\Pi_i A_i$ is an open set of $\mathbb{R}^n$.

   (b) Let $(A_i)_{i \in [\![1,n]\!]}$ be n sets of $\mathbb{R}$, with $n \in \mathbb{N}^*$, such that $\Pi_i A_i$ is an open set of $\mathbb{R}^n$. Show by induction that the (A_i) are open in $\mathbb{R}$.

4. (a) Let A be an open set of R^n and B an open set of $\mathbb{R}^p$, with $(n, p) \in \mathbb{N}^2$. Show that $A \times B$ is an open set of $\mathbb{R}^{n+p}$.

   (b) Let A be a set of R^n and B a set of $\mathbb{R}^p$, with $(n, p) \in \mathbb{N}^2$, such that $A \times B$ is an open set of $\mathbb{R}^{n+p}$. Show that A and B are respectively open in $\mathbb{R}^n$ and $\mathbb{R}^p$.

5. (a) Let $(A_i)_{i \in [\![1,n]\!]}$ be n open sets, each in $\mathbb{R}^{p_i}$, where $p_i \in \mathbb{N}^*$ is specific to A_i. Show by induction that $\Pi_i A_i$ is an open set of $\mathbb{R}^{\sum p_i}$.

   (b) Let $(A_i)_{i \in [\![1,n]\!]}$ be n sets, each in $\mathbb{R}^{p_i}$, where $p_i \in \mathbb{N}^*$ is specific to A_i, and such that $\Pi_i A_i$ is an open set of $\mathbb{R}^{\sum p_i}$. Show by induction that each A_i is open in $\mathbb{R}^{p_i}$.

6. (a) Let $(A_i)_{i \in [\![1,n]\!]}$ be n closed sets, each in $\mathbb{R}^{p_i}$, where $p_i \in \mathbb{N}^*$ is specific to A_i. Show that $\Pi_i A_i$ is a closed set of $\mathbb{R}^{\sum p_i}$.

   (b) Let $(A_i)_{i \in [\![1,n]\!]}$ be n sets, each in $\mathbb{R}^{p_i}$, where $p_i \in \mathbb{N}^*$ is specific to A_i, and such that $\Pi_i A_i$ is a closed set of $\mathbb{R}^{\sum p_i}$. Show that each A_i is closed in $\mathbb{R}^{p_i}$.

7. (a) Let $(A_i)_{i \in [\![1,n]\!]}$ be n compact sets, each in $\mathbb{R}^{p_i}$, where $p_i \in \mathbb{N}^*$ is specific to A_i. Show that $\Pi_i A_i$ is a compact set of $\mathbb{R}^{\sum p_i}$. *Hint: draw an example in* $\mathbb{R}^2$.

   (b) Let $(A_i)_{i \in [\![1,n]\!]}$ be n sets, each in $\mathbb{R}^{p_i}$, where $p_i \in \mathbb{N}^*$ is specific to A_i, and such that $\Pi_i A_i$ is a compact set of $\mathbb{R}^{\sum p_i}$. Show that each A_i is compact in $\mathbb{R}^{p_i}$.